The harmonic oscillator, dimensional analysis and inflationary solutions

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In this work, focused on the production of exact inflationary solutions using dimensional analysis, it is shown how to explain inflation from a pragmatic and basic point of view, in a step-by-step process, starting from the one-dimensional harmonic oscillator.

I. INTRODUCTION

"No discussion of modern cosmology would be complete without some mention of inflation .... This is the idea that ... there existed a phase in which the universe expands much faster (in fact exponentially) than the rate given by the standard scenario"[1]. Although now more than 20 years old, this idea survived well in the modern cosmology: "cosmologists ... fell in love with inflation, because it could explain some nagging problems raised by the standard big bang model"[2]. In fact, inflation is now probably more alive than ever, since the most recent measurements of the cosmic microwave background indicate an almost flat universe, a result *predicted* by inflation. But despite the success of inflation among cosmologists, and several attempts to explain it, inflation is only a general idea, since no one came up with a definitive inflationary scenario.

As an important part of the modern cosmology, inflation is now a subject present in several introductory texts, articles and books. In one science dictionary, for example, one can read, under the entry "early universe", that "an important idea in the theory of the early universe is that of *inflation* – the idea that the nature of the vacuum state gave rise, after the big bang, to an exponential expansion of the universe"[3]. This shows that is common to associate the inflationary idea with a 'decay' or 'phase transition' of the vacuum (or something alike), and it is almost canonical to present inflation in this way[4].

To be more specific, the most usual approach is to show inflation as an effect due to the dynamics of a scalar field, with use of an approximation known as '*slow rollover'*. However, from a practical point of view, it is possible to present the construction of inflationary solutions without the mention or use of any approximations, from very basic concepts. The dynamics of one single homogeneous scalar field is one-dimensional, and as such it has similarities with the dynamics of the one-dimensional harmonic oscillator, and this fact can be used in an introductory approach to the idea of inflation, together with 'suggestions' given by dimensional analysis.

It is the purpose of this work, then, to show how to explain inflation from a pragmatic and basic point of view, in a step-by-step process, starting from the harmonic oscillator to finally get, after some general mathematical efforts, *exact* inflationary solutions. Therefore, it must be noted that here the focus is on the production of inflationary solutions, and not on interpreting what they can mean or what kind of problems they can solve.

It is important to notice that, though pedagogical, this text takes advantage of common conventions of the scientific notation. For example, throughout this text natural units are used, ie, $c = G = \hbar = 1$, unless stated otherwise, like in the section of dimensional analysis. Also, the *sum convention* is always present, ie, terms like $\sum_{\mu=0}^{3} (x^\mu)^2$ are written as $x^\mu x_\mu$, with greek indices $(\mu, \nu, \ldots)$ running from 0 to 3 and latin ones $(i, j, \ldots)$ from 1 to 3. Finally, generic constants are written as c_i $(c_1, c_2, \ldots)$ and λ_i $(\lambda_1, \lambda_2, \ldots)$.

II. THE HARMONIC OSCILLATOR

A basic *one-dimensional* system may be constructed with a particle of mass m and velocity v subject to a force F, from which one obtains a *potential energy* $V(x)$, so that $V(x) = -\int F dx$. Such system may be described by the *Lagrangian*

$$L(x, v) = T - V(x) = \frac{mv^2}{2} - V(x), \qquad (1)$$

where T is the kinetic energy of the particle. Notice that the Lagrangian is a function of the coordinate x and its first derivative, the velocity v, ie, $L = L(x, v)$.

From the Lagrangian one can build another function, the *Hamiltonian*

$$H \equiv pv - L = \frac{p^2}{2m} + V(x), \qquad (2)$$

where

$$p \equiv \frac{\partial L}{\partial v} = mv \qquad (3)$$

is the momentum associated with the coordinate x. The Hamiltonian, which is a function of x and p, represents the total energy of the system.

If the total energy of the system is conserved, then

$$\frac{dH}{dt} = \frac{\partial H}{\partial x}\left(\frac{dx}{dt}\right) + \frac{\partial H}{\partial p}\left(\frac{dp}{dt}\right) + \frac{\partial H}{\partial t} = 0, \qquad (4)$$

or, equivalently,

$$\frac{d}{dt}\left(\frac{mv^2}{2}+V\right)=0, \tag{5}$$

from where one gets the equation of movement

$$m\frac{d^2x}{dt^2}+\frac{dV}{dx}=0. \tag{6}$$

What kind of potential can one have in this equation? *Possible* answers can be given by dimensional analysis. Noting that

$$[V]=kg^1m^2s^{-2} \tag{7}$$

where $[Q]$ means the dimension of the quantity Q, and kg, m and s are the units of mass, distance and time in SI units, respectively, relations like

$$V=\lambda_1 x^p \tag{8}$$

are possible only if

$$[\lambda_1]=\left[Vx^{-p}\right]=kg^1m^{2-p}s^{-2}. \tag{9}$$

Therefore, in this case the system must have some measurable constant λ_1 with dimensions given by (9).

"The most important problem in one-dimensional motion ... is the harmonic oscillator"[5]. This system may be represented by a mass fastened to a spring. The spring is characterized by a constant k which has

$$[k]=kg^1s^{-2}, \tag{10}$$

so that, for the harmonic oscillator, the correct form of the potential is

$$V(x)=\lambda_1x^2=\mu kx^2, \tag{11}$$

where μ is a dimensionless constant. Since the spring obeys Hooke's law $F=-kx$, then $\mu=1/2$, and the equation of movement for the harmonic oscillator is

$$m\frac{d^2x}{dt^2}+kx=0. \tag{12}$$

The one-dimensional harmonic oscillator, which by its simplicity represents a prototype for many one-dimensional systems, can be generalized in the context of a field theory in a 4-dimensional – or *covariant* – formalism. Such generalization is the main topic of the next section, where the Lagrangian formulation is also written covariantly.

III. A GENERALIZED HARMONIC OSCILLATOR

For a covariant generalization of the harmonic oscillator one needs some new concepts, usually presented as a 'preface to curvature'[6]:

- the idea of a space-time, with coordinates x^μ and the metric $g_{\mu\nu}$ (of determinant g), described by the interval $ds^2\equiv g_{\mu\nu}dx^\mu dx^\nu$;
- the covariant derivative ∇_μ, which substitutes the usual derivative $\partial_\mu\equiv d/dx^\mu$, and related to it by a conexion $\Gamma^\alpha_{\beta\gamma}$, such that $(\nabla_\mu-\partial_\mu)\,x^\nu=\Gamma^\nu_{\mu\alpha}x^\alpha$ *and* $\nabla_\mu g_{\alpha\beta}=0$;
- the idea of a *Lagrangian density* $\mathcal{L}=\mathcal{L}\left(x^\mu,\nabla_\mu\varphi_i\right)$, which is a *scalar density*[7] of weight $+1$, and which depends on the space-time coordinates x^μ through *fields* $\varphi_i\left(x^\mu\right)$ and their gradients $\nabla_\mu\varphi_i\left(x^\mu\right)$.

Using these concepts, the simplest covariant field generalization of the harmonic oscillator is

$$\mathcal{L}=\sqrt{-g}\left[\frac{1}{2}g^{\mu\nu}\nabla_\mu\varphi\nabla_\nu\varphi-V(\varphi)\right]. \tag{13}$$

Now, for *any* infinite system described by a general Lagrangian density $\mathcal{L}$, one can write an *energy-momentum tensor*[8]

$$T^{\mu\nu}\equiv\frac{1}{\sqrt{-g}}\left[\pi^\mu\nabla^\nu\varphi_i-g^{\mu\nu}\mathcal{L}\right], \tag{14}$$

where

$$\pi^\mu\equiv\frac{\partial\mathcal{L}}{\partial\left(\nabla_\mu\varphi\right)}, \tag{15}$$

and from it, by use of the conservation condition

$$\nabla_\mu T^{\mu\nu}=0, \tag{16}$$

to obtain the equations of movement – the *field equations* – of the system.

In the case of the generalized harmonic oscillator the field equation obtained is the Klein-Gordon equation,

$$\Box\varphi+\frac{dV}{d\varphi}\equiv g^{\mu\nu}\nabla_\mu\nabla_\nu\varphi+\frac{dV}{d\varphi}=0, \tag{17}$$

where $\Box$ is the d'Alembertian operator. Notice that both the covariant derivative *and* the d'Alembertian depend on the metric $g_{\mu\nu}$, and that one can write for a scalar, explicitly[9],

$$\Box\varphi=[-g]^{-1/2}\,\partial_\mu\left(\sqrt{-g}g^{\mu\nu}\partial_\nu\varphi\right). \tag{18}$$

Everything done in this section has a good analogy with the classical description of the one-dimensional harmonic oscillator of the previous section. For example, the one-dimensional harmonic oscillator has as equation of movement a linear second-order differential equation, while the field equation for the generalized harmonic oscillator is a generalization of this, with the d'Alembertian operator acting as the second-order differential operator. However, the d'Alembertian depends on the metric $g_{\mu\nu}$ of the spacetime. So, in order to solve the field equation for the generalized harmonic oscillator first one must find how to write it explicitly. This is done in the next section with the geometric formalism of General Relativity.

IV. GENERAL RELATIVITY

The relation between the metric of a space-time and its content of matter *and* energy is given by the Einstein Equations

$$G_{\mu\nu} \equiv R_{\mu\nu} - \frac{1}{2} g_{\mu\nu} R = 8\pi T_{\mu\nu} + g_{\mu\nu}\Lambda, \tag{19}$$

where $R_{\mu\nu}$, the Ricci tensor, is a geometrical quantity built with the metric and its derivatives up to second order, $R \equiv g^{\mu\nu} R_{\mu\nu}$ is the Ricci scalar, and Λ is the cosmological constant. If one wants a very general solution of Einstein Equations, representing an isotropic and homogeneous space-time, it is common to use the metric present in the Friedmann–Lemaître-Robertson-Walker (FLRW) element of line

$$ds^2 = dt^2 - a^2(t)\left[\frac{dr^2}{1-kr^2} + r^2\left(d\theta^2 + \sin^2\theta d\phi^2\right)\right], \tag{20}$$

where $a(t)$ is the *scale factor*, and where $k = 0, \pm 1$ is a number, the *parameter of curvature*. Another important assumption is that the matter content of the universe may be represented by a generic perfect fluid with

$$T^{\mu\nu} = (\rho + p) u^\mu u^\nu - p g^{\mu\nu} \tag{21}$$

as energy-momentum tensor, where u^μ is the 4-velocity of the fluid, ρ its energy density and p its pressure. Using these ingredients, one obtains two equations, one relating ρ and a,

$$\frac{1}{a^2}\left(\frac{da}{dt}\right)^2 + \frac{k}{a^2} = \frac{8\pi}{3}\rho + \frac{\Lambda}{3}, \tag{22}$$

known as Friedmann's Equation, and another relating p and a,

$$\frac{2}{a}\frac{d^2a}{dt^2} + \frac{1}{a^2}\left(\frac{da}{dt}\right)^2 + \frac{k}{a^2} = \Lambda - 8\pi p. \tag{23}$$

It is important to notice that these two equations are not independent, due to the conservation of energy,

$$d(\rho V) + p dV = 0, \tag{24}$$

which relates the variation of ρ with p and the variation of the volume V of the fluid.

The important thing to be noticed here is that the metric $g_{\mu\nu}$ has a time dependence, in the scale factor $a(t)$, represented by any of the equations (22) and (23). Therefore, different contents of matter and energy, expressed in ρ and p, give birth to different temporal evolutions of the metric. One kind of these possible evolutions received the name of inflation.

V. THE FIELD EQUATIONS

The FLRW metric also allows to write explicitly the d'Alembertian, through eq. (18), such that the field equation becomes

$$\frac{\partial^2\varphi}{\partial t^2} + \frac{3}{a}\frac{da}{dt}\frac{\partial\varphi}{\partial t} + \frac{1}{a^2}\nabla^2\varphi + \frac{dV}{d\varphi} = 0, \tag{25}$$

where ∇^2 is the usual three-dimensional Laplacian operator.

Can the energy-momentum tensor of the scalar field, obtained from equations (13) and (14), be compared to the one of a perfect fluid? If one assumes that the field is homogeneous there is no spatial gradients and, therefore, from the component T^{00}, one gets

$$\rho = \frac{1}{2}\left(\frac{d\varphi}{dt}\right)^2 + V(\varphi) \tag{26}$$

and, after this, it becomes clear that

$$p = \frac{1}{2}\left(\frac{d\varphi}{dt}\right)^2 - V(\varphi). \tag{27}$$

This gives the *equation of state*

$$p = w\rho, \tag{28}$$

where

$$w \equiv \left[\frac{1}{2}\left(\frac{d\varphi}{dt}\right)^2 - V(\varphi)\right]\left[\frac{1}{2}\left(\frac{d\varphi}{dt}\right)^2 + V(\varphi)\right]^{-1}. \tag{29}$$

Using the above results one obtains two independent equations,

$$\left(\frac{1}{a}\frac{da}{dt}\right)^2 + \frac{k}{a^2} = \frac{8\pi}{3}\left[\frac{1}{2}\left(\frac{d\varphi}{dt}\right)^2 + V(\varphi)\right] + \frac{\Lambda}{3}, \tag{30}$$

or its equivalent

$$\frac{2}{a}\frac{d^2a}{dt^2} + \frac{1}{a^2}\left(\frac{da}{dt}\right)^2 + \frac{k}{a^2} = \Lambda - 8\pi\left[\frac{1}{2}\left(\frac{d\varphi}{dt}\right)^2 - V(\varphi)\right], \tag{31}$$

and

$$\frac{d^2\varphi}{dt^2} + \frac{3}{a}\frac{da}{dt}\frac{d\varphi}{dt} + \frac{dV}{d\varphi} = 0, \tag{32}$$

which becomes the equation for the harmonic oscillator if a is constant, or its equivalent

$$\frac{d}{dt}\left[\frac{1}{2}\left(\frac{d\varphi}{dt}\right)^2 + V(\varphi)\right] + \frac{3}{a}\frac{da}{dt}\left(\frac{d\varphi}{dt}\right)^2 = 0, \tag{33}$$

which is nothing more than the condition for energy conservation, with a^3 assuming the role of the volume.

Unfortunately, the *two* independent equations obtained contain *three* unknowns, a, φ and V. Therefore, in order to solve them one needs something else, like simplifying assumptions, or an equation relating V and φ, for example. The most common procedure is to specify some kind of potential, $V(\varphi)$, and from it solve the equations for the field and the scale factor. In the next section it is shown that a group of inflationary solutions can be built with relations suggested by dimensional analysis.

VI. DIMENSIONAL CONSIDERATIONS

A natural choice to solve the system formed by the equations (30) and (32) consists in using as an extra hypothesis some relation between the unknowns a and φ (and their derivatives) suggested by dimensional analysis. As a specific example, one can use only relations where all needed constants are built with a combination of the universal constants c and G, both present in General Relativity.

The first step consists in observing that

$$\left[G^p c^q\right] = kg^{-p} m^{3p+q} s^{-2p-q}. \tag{34}$$

Assuming

$$[a] = m \tag{35}$$

and

$$\left[\left(\frac{d\varphi}{dt}\right)^2\right] = kg^1 m^{-1} s^{-2}, \tag{36}$$

one can 'try' relations such as

$$\left(\frac{d\varphi}{dt}\right)^{2k} = \lambda_1 a^{-\ell}, \tag{37}$$

where

$$[\lambda_1] = \left[\left(\frac{d\varphi}{dt}\right)^{2k} a^{\ell}\right] = kg^k m^{-k+\ell} s^{-2k}. \tag{38}$$

Comparing this last result with equation (34) one obtains the system of equations

$$\begin{cases} k = -p \\ -k + \ell = 3p + q \\ -2k = -2p - q \end{cases}, \tag{39}$$

which gives $p = -k$, $q = 4k$ and $\ell = 2k$. Choosing $k = 1/2$ one gets $[\lambda_1] = \left[G^{-1/2} c^2\right]$ and the relation

$$\frac{d\varphi}{dt} = \lambda_1 a^{-1}. \tag{40}$$

Substituting this 'ansatz' in (30) and in (33), one obtains a new system formed by the equations

$$\left(\frac{1}{a}\frac{da}{dt}\right)^2 + \frac{k}{a^2} = \frac{8\pi}{3}\left[\frac{\lambda_1^2}{2a^2} + V(\varphi)\right] + \frac{\Lambda}{3} \tag{41}$$

and

$$\frac{2\lambda_1^2}{a^3}\frac{da}{dt} + \frac{dV}{dt} = 0. \tag{42}$$

This last equation shows that

$$V = V_0 + \lambda_1^2 a^{-2}, \tag{43}$$

with V_0 being a constant, what finally allows one the obtain the solutions

$$a(t) = \frac{1}{2\alpha}\left[c_1 e^{\alpha t} + \frac{k - 4\pi\lambda_1^2}{c_1} e^{-\alpha t}\right], \tag{44}$$

$$\varphi(t) = \varphi_0 + \frac{\lambda_1}{\left(4\pi\lambda_1^2 - k\right)^{1/2}} \ln\left[\frac{\left(4\pi\lambda_1^2 - k\right)^{1/2} - c_1 e^{\alpha t}}{\left(4\pi\lambda_1^2 - k\right)^{1/2} + c_1 e^{\alpha t}}\right], \tag{45}$$

and

$$V(\varphi) = V_0 - \frac{\alpha^2}{k - 4\pi\lambda_1^2}\sinh^2\left[\frac{\left(4\pi\lambda_1^2 - k\right)^{1/2}}{\lambda_1}(\varphi - \varphi_0)\right], \tag{46}$$

where c_1, φ_0 and $3\alpha^2 \equiv 8\pi V_0 + \Lambda$ are constants. Notice that these solutions, which do not always include the initial condition $a(0) = 0$, are *not* valid for the case $\alpha = 0$, when one has instead

$$a(t) = \left(4\pi\lambda_1^2 - k\right)^{1/2} t, \tag{47}$$

$$\varphi(t) = \varphi_0 + \lambda_1\left(4\pi\lambda_1^2 - k\right)^{-1/2} \ln t, \tag{48}$$

and

$$V(\varphi) = -\frac{\Lambda}{8\pi} - \frac{\lambda_1^2}{k - 4\pi\lambda_1^2}\exp\left[-2\sqrt{4\pi - \frac{k}{\lambda_1^2}}(\varphi - \varphi_0)\right]. \tag{49}$$

Another possible relation inspired by dimensional analysis is

$$\left(\frac{d\varphi}{dt}\right)^{2k} = \lambda_2 H^n \equiv \lambda_2\left(\frac{1}{a}\frac{da}{dt}\right)^n, \tag{50}$$

with H being now the *Hubble function*, and

$$[\lambda_2] = \left[\left(\frac{d\varphi}{dt}\right)^{2k} H^{-n}\right] = kg^k m^{-k} s^{-2k+n}. \tag{51}$$

Using (34) one has, then, $n = q = -2p = 2k$. Choosing $k = 1/2$, $[\lambda_2] = \left[G^{-1/2} c^1\right]$ and the *ansatz* now becomes

$$\frac{d\varphi}{dt} = \lambda_2 \frac{1}{a}\frac{da}{dt} = \lambda_2 \frac{d\ln a}{dt}, \tag{52}$$

or

$$a = \exp\left(\frac{\varphi - \varphi_0}{\lambda_2}\right), \tag{53}$$

what substituted in (30) and in (33) yields, for $k = \Lambda = 0$,

$$a(t) = t^{1/\left(4\pi\lambda_2^2\right)}, \tag{54}$$

$$\varphi(t) = \varphi_0 + \frac{1}{4\pi\lambda_2}\ln t, \tag{55}$$

and

$$V(\varphi) = \left(\frac{3}{8\pi\lambda_2^2} - \frac{1}{2}\right)\frac{\exp\left[-8\pi\lambda_2(\varphi-\varphi_0)\right]}{16\pi^2\lambda_2^2}, \qquad (56)$$

where φ_0 is a constant.

A last example is given by the relation

$$\left(\frac{d\varphi}{dt}\right)^{2k} = \lambda_3\left(\frac{dH}{dt}\right)^{j}, \qquad (57)$$

where

$$[\lambda_3] = \left[\left(\frac{d\varphi}{dt}\right)^{2k}\left(\frac{dH}{dt}\right)^{-j}\right] = kg^k m^{-k} s^{-2k+2j}. \qquad (58)$$

Now, the indices are such that $2j = q = -2p = 2k$, and choosing $k = 1$ one gets $[\lambda_3] = \left[G^{-1}c^2\right]$ and the relation

$$\left(\frac{d\varphi}{dt}\right)^{2} = \lambda_3\frac{dH}{dt}, \qquad (59)$$

with the solutions

$$a(t) = \frac{1}{2\beta}\left[c_2 e^{\beta t} + \frac{1}{c_2}\left(\frac{k}{1+4\pi\lambda_3}\right)e^{-\beta t}\right], \qquad (60)$$

$$\varphi(t) = \varphi_0 + \sqrt{-\lambda_3}\ln\left[\frac{k^{1/2}(1+4\pi\lambda_3)^{-1/2} - c_2 e^{\alpha t}}{k^{1/2}(1+4\pi\lambda_3)^{-1/2} + c_2 e^{\alpha t}}\right], \qquad (61)$$

and

$$V(\varphi) = V_0 - \lambda_3\beta^2\left(\frac{1}{2} + \cosh^2\frac{\varphi-\varphi_0}{\sqrt{-\lambda_3}}\right), \qquad (62)$$

where c_2, V_0 and φ_0 are constants, and $3\beta^2 \equiv (8\pi V_0 + \Lambda)(1+4\pi\lambda_3)^{-1}$. These solutions, which do not always include the initial condition $a(0) = 0$, are valid for $\beta \neq 0$. When $\beta = 0$ one has instead

$$a(t) = k^{1/2}(1+4\pi\lambda_3)^{-1/2}\,t, \qquad (63)$$

$$\varphi(t) = \varphi_0 + \sqrt{-\lambda_3}\ln t, \qquad (64)$$

and

$$V(\varphi) = -\frac{\Lambda}{8\pi} - \lambda_3\exp\left[-2\frac{(\varphi-\varphi_0)}{\sqrt{-\lambda_3}}\right]. \qquad (65)$$

It is interesting to notice that a relation like

$$\left(\frac{d\varphi}{dt}\right)^{2k} = \lambda_4\left(\frac{da}{dt}\right)^{q} \qquad (66)$$

is not valid if one wants λ_4 a constant with dimension equal to the dimension of a combination of only c and G.

To finish this section it is important to remark that the solutions obtained above, all of them generalizations of solutions presented before in the literature[10], are only of two kinds with relation to the behaviour of the scale factor a: a power function of t and a combination of exponentials of t. In the next section it will be discussed if such kinds of solutions are inflationary solutions.

VII. INFLATION

The inflationary epoch of the universe is usually seen as the age where the universe inflated very fast, in an exponential way. However, this 'definition' of inflation can be widened: one can say that "inflation is defined as an epoch in the history of the universe during which the cosmic expansion is accelerated"[11]. This means that a solution of Friedmann's equation for the scale factor $a(t)$ can be seen as an inflationary solution if it is an expanding solution *and* its second time derivative is positive,

$$\frac{d^2a}{dt^2} > 0. \qquad (67)$$

However, combining (22) and (23), one can see that

$$\frac{d^2a}{dt^2} = \frac{a}{3}\left[\Lambda - 4\pi(\rho+3p)\right], \qquad (68)$$

so that, considering only solutions and times where $a \geq 0$, the condition for inflation becomes equivalent to have

$$\Lambda > 4\pi(\rho+3p). \qquad (69)$$

Defining a new density ρ_Λ such that $8\pi\rho_\Lambda \equiv \Lambda$ and assuming the equation of state $p = w\rho$ this relation is written as

$$\frac{\rho_\Lambda}{\rho} > \frac{1+3w}{2}. \qquad (70)$$

For ordinary matter ('dust') $w = 0$, while for radiation $w = 1/3$. If, however, $\Lambda = 0$ then one must necessarily have $w < -1/3$.

The solutions presented in the previous section are of two kinds: a power function of t, where

$$a(t) = c_1 t^\lambda \Rightarrow \frac{d^2a}{dt^2} = \frac{\lambda(\lambda-1)}{t^2}a(t), \qquad (71)$$

and a combination of exponentials of t, with

$$a(t) = c_2 e^{\lambda t} + c_3 e^{-\lambda t} \Rightarrow \frac{d^2a}{dt^2} = \lambda^2 a(t). \qquad (72)$$

Clearly, such solutions obey the condition for inflation with two restrictions: $\lambda > 1$, in the power function case, and $\lambda^2 > 0$, in the exponential case. Therefore, from all solutions presented here, in this section, only the ones given by equations (47) and (63) can not be considered inflationary, while the solution given by eq. (44) is inflationary for $\Lambda > -8\pi V_0$, the one given by eq. (54) is inflationary for $1 > 4\pi\lambda_2^2$, and the one given by (60) is inflationary wherever $(8\pi V_0 + \Lambda)(1+4\pi\lambda_3)^{-1} > 0$.

VIII. FINAL REMARKS

The purpose of this work was to show, using as example the 'idea' of inflation, that even very basic concepts, such

as the use of dimensional analysis, can have its role in the search of more sophisticated results. This process can be illustrated a little more with one last example. If one tries the relation

$$\left(\frac{d\varphi}{dt}\right)^{2k} = \lambda_2 H^n + \lambda_2', \tag{73}$$

one gets that

$$[\lambda_2'] = kg^k m^{-k} s^{-2k}, \tag{74}$$

and certainly this can not be obtained using only c and G. Now if one allows the use of Planck's constant $\hbar$, one has

$$[G^p c^q \hbar^r] = kg^{-p+r} m^{3p+q+2r} s^{-2p-q-r}, \tag{75}$$

such that $2r = p = -2q/7 = -2k$. Choosing $k = 1$, one obtains $[\lambda_2'] = \left[G^{-2} c^8 (\hbar c)^{-1}\right]$ and

$$\left(\frac{d\varphi}{dt}\right)^{2} = \lambda_2 H^2 + \lambda_2' \tag{76}$$

Deriving this relation and substituting the result in (33) one gets

$$\lambda_2 H \frac{dH}{dt} + 3H\left(\lambda_2 H^2 + \lambda_2'\right) + \frac{dV}{dt} = 0. \tag{77}$$

Now, inserting (76) in (30) and returning the result into (77) one gets

$$\frac{dH}{dt} + 4\pi\left(\lambda_2 H^2 + \lambda_2'\right) - \frac{k}{a^2} = 0. \tag{78}$$

This equation is easily solved when $k = 0$, with the results

$$H(t) = -\sqrt{\frac{\lambda_2'}{\lambda_2}} \tan\left[4\pi\sqrt{\lambda_2\lambda_2'}\,(t - t_0)\right] \tag{79}$$

and

$$a(t) = a_0 \cos^{1/(4\pi\lambda_2)}\left[4\pi\sqrt{\lambda_2\lambda_2'}\,(t - t_0)\right], \tag{80}$$

where t_0 and a_0 are constants of integration. Imposing that $a(0) = 0$ one gets $2\gamma t_0 = \pi$, and if $\lambda_2 > 0$ and $\lambda_2' < 0$, then $\sqrt{\lambda_2\lambda_2'}$ is a purely imaginary quantity, such that

$$a(t) = a_1 \sinh^{1/(4\pi\lambda_2)} 4\pi\sqrt{\lambda_2\,|\lambda_2'|}t, \tag{81}$$

where a_1 is a real positive constant. Finally, if one chooses $4\pi\lambda_2 = 3$, then, from (30) and (76), $V = V_0 = (4\pi\,|\lambda_2'| - \Lambda)\,(8\pi)^{-1}$ and

$$a(t) = a_1 \sinh^{1/3} \sqrt{3\,(8\pi V_0 + \Lambda)}t. \tag{82}$$

Notice that this *exact* solution, obtained previously in the literature through the supposition of a specific form for the potential V of the scalar field[11], is just a particular case of a more general solution given by eq. (81).

It is important to emphasize that if one uses dimensional analysis to obtain new reasonable relations between physical quantities, relations which can be used as working hypothesis, one must worry about the validity – or origin – of such relations: do they represent something more than a possible mathematical relation with the dimensions adjusted correctly? For example, in the case worked above does the use of Planck's constant $\hbar$ to obtain the right dimensions of the constant λ_2' indicate that such constant must have a quantum origin?

However, putting aside interpretative questions, one message this work wants to leave is that dimensional analysis is a very helpful and handy tool, which beyond allowing to check the consistency of some results, can also suggest new approaches, explanations and results, both in teaching and research.

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